

# A new proof of the cohomological criterion for noetherian regular local rings

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## Abstract

Let  $(A, \mathfrak{m}, k = A/\mathfrak{m})$  be a noetherian local ring. Then it is equivalent  $n = \dim A = \dim_k \mathfrak{m}/\mathfrak{m}^2$  and  $\mathrm{Tor}_i^A(k, k) = 0$  for all  $i \gg 0$ . The article gives a proof with the change-of-ring spectral sequence.

## 1 Theorem and Proof

### 1.1 Acknowledgements

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### 1.2 Auxiliary Results

We start with some results used later in the proof.

**Lemma 1.1** *Let  $(A, \mathfrak{m})$  be a local ring and  $L_\bullet \rightarrow M \rightarrow 0$  a free  $A$ -resolution of a finitely generated  $A$ -module  $M$ . Then  $(L_\bullet, d_\bullet)$  can be chosen such, that  $d_i(L_i) \subseteq \mathfrak{m}L_{i-1}$  holds.*

**Proof.** Assume  $L_{i-1} \rightarrow L_{i-2} \rightarrow \cdots \rightarrow L_0 \rightarrow M$  already constructed and  $0 \rightarrow Z \rightarrow L_{i-1} \rightarrow L_{i-2}$  exact, as well as  $L_{i-1}$  with minimal rank. Then it is  $Z \subseteq \mathfrak{m}L_{i-1}$ .

Write  $L_{i-1}$  as  $L_{i-1} = Ae_1 \oplus \cdots \oplus Ae_r$  and let  $z = a_1e_1 + \cdots + a_re_r \in Z$ . If there is  $a_1 \notin \mathfrak{m}$ , therefore  $a_1 \in A^*$ , one can assume  $a_1 = 1$  without restriction of generality. So we have the equation  $d_{i-1}(e_1) = -a_2d_{i-1}(e_2) - \cdots - a_rd_{i-1}(e_r)$

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and the summand  $Ae_1$  in  $L_{i-1}$  would be superfluous and one could chose  $L_{i-1}$  with smaller rank.

At the next step one choses  $L_i$  with minimal rank and a surjective map  $L_i \rightarrow Z \rightarrow 0$  and proceeds with the construction inductively.

The following definition is from Godement [2][4.2, p. 77]:

**Definition 1.1** *Let  $F$  be a filtered differential module with derivation  $d : F \rightarrow F$  with  $d^2 = 0$  and filtration  $(F^p)$  where  $F^{p+1} \subseteq F^p$  and  $d(F^p) \subseteq F^p$  holds.*

*Then the associated spectral sequence is*

$$E_r^p = (Z_r^p) / (dZ_{r-1}^{p-r+1} + Z_{r-1}^{p+1}) \quad (1)$$

where  $Z_r^p = \{x \in F^p \mid dx \in F^{p+r}\}$ .

*In the special case  $r = 2$  we have*

$$E_2^p = (Z_2^p) / (dZ_1^{p-1} + Z_1^{p+1}) \quad (2)$$

**Definition 1.2** *Let  $A$  be a commutative ring. Then  $\text{gldh } A = \sup\{p \mid \text{Ext}_A^p(M, N) \neq 0\}$ , where  $M, N$  are arbitrary  $A$ -Modules, is called the global dimension of  $A$ .*

The following proposition follows from Serre [3][IV - 35, Corollaire 2]

**Proposition 1.1** *For a noetherian local ring  $(A, \mathfrak{m}, k = A/\mathfrak{m})$  we have the equivalence*

- a) *It is  $\text{gldh } A \leq n$ .*
- b) *It is  $\text{Tor}_{n+1}^A(k, k) = 0$*

From Serre [3][IV - 35, Proposition 21] we cite

**Proposition 1.2 (Auslander-Buchsbaum-Formula)** *Let  $(A, \mathfrak{m})$  be a local noetherian ring with finite global dimension  $\text{gldh } A = n$ . Furthermore let  $M$  be a finitely generated  $A$ -Module. Then we have*

$$\text{pd } M + \text{depth } M = n = \text{gldh } A \quad (3)$$

We use this formula several times in the proof without explicitly referring to it.

### 1.3 Main Theorem

**Proposition 1.3** *Let  $(A, \mathfrak{m})$  be a noetherian local ring of dimension  $n$ . Then the following conditions on  $A$  are equivalent:*

- a) *The homological dimension  $\text{gldh } A$  is finite. In this case it holds automatically that  $n = \text{gldh } A = \dim A$ .*
- b) *It is  $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = n$*
- c)  *$\mathfrak{m}$  can be generated by  $n$  elements.*
- d) *The tangential cone is  $\text{gr}_{\mathfrak{m}} A \cong k[X_1, \dots, X_n]$*

where  $k = A/\mathfrak{m}$  is the residue field of  $A$ .

**Proof.** We will call a ring that fulfills b), c) oder d) *geometrically regular*. A ring that fulfills a) we call *cohomologically regular*.

First, because of the Nakayama–Lemma and because  $\mathfrak{m}$  must be generated by at least  $n$  elements, b) is equivalent to c). Furthermore we have then a surjection  $k[T_1, \dots, T_n] \twoheadrightarrow \text{gr}_{\mathfrak{m}} A$ , therefore an isomorphism  $k[T_1, \dots, T_n]/I \xrightarrow{\sim} \text{gr}_{\mathfrak{m}} A$ . As  $n = \dim A = \dim \text{gr}_{\mathfrak{m}} A$ , it must be  $I = 0$ . Thereby d) is proved.

Now let d) hold: If  $\bar{x}_i$  is the image of  $T_i$  in  $\mathfrak{m}/\mathfrak{m}^2$  and  $x_i \in \mathfrak{m}$  a preimage, then  $x_1, \dots, x_n$  is a regular sequence in  $A$ . Therefore  $K_{\bullet}(x_1, \dots, x_n) \rightarrow k \rightarrow 0$ , the Koszul–complex for the  $x_i$ , is a free resolution of length  $n$  of  $k$ . So we have  $\text{Tor}_j(k, M) = 0$  for  $j > n$  and  $\text{Tor}_n(k, k) = k \neq 0$ . This shows  $A$  is cohomologically regular with  $\text{gldh } A = n$ .

Now let, for the reverse direction,  $(A, \mathfrak{m})$  be a local noetherian ring with  $\text{gldh } A < \infty$ . We show by induction over  $\dim A$ , that  $A$  is then a geometrically regular ring too.

First for  $\dim A = 0$  the ring  $A$  is an artinian ring and  $\text{depth } A = 0$ . So we have  $\text{gldh } A = \text{depth } A + \text{pd}_A A = 0$  and especially  $\dim A = \text{gldh } A = 0$ .

It follows that  $\text{pd}_A k = 0$  so that  $k$  is projective, therefore free, so that  $k = A^r$ . With  $-\otimes_A k$  it follows  $r = 1$ , so  $A = k$  and  $\mathfrak{m} = 0$ . This closes the case  $\dim A = 0$ .

Now let  $\dim A = n$  and assume the theorem already proven for  $\dim A < n$ . Choose a prime ideal  $\mathfrak{p} \subseteq \mathfrak{m}$  with  $\dim A_{\mathfrak{p}} = n - 1$  and consider the free  $A$ –resolution

$$F_{\bullet} \rightarrow A/\mathfrak{p} \rightarrow 0.$$

It is of finite length. Tensoring by  $A_{\mathfrak{p}}$  gives a finite free  $A_{\mathfrak{p}}$ –resolution of  $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ . Therefore we have  $\text{Tor}_{\nu}^{A_{\mathfrak{p}}}(k(\mathfrak{p}), k(\mathfrak{p})) = 0$  for all  $\nu \gg 0$ . So it is  $\text{gldh } A_{\mathfrak{p}} < \infty$  and by the inductive assumption  $\text{gldh } A_{\mathfrak{p}} = \dim A_{\mathfrak{p}} = n - 1$ . The shortest free resolution

$$F_{\bullet} \rightarrow A/\mathfrak{p} \rightarrow 0$$

has therefore at least the length  $r \geq n - 1$ . Now it is  $\text{pd } A/\mathfrak{p} + \text{depth } A/\mathfrak{p} = \text{gldh } A$ . Also we have  $\text{depth } A/\mathfrak{p} \geq 1$ . Therefore  $\text{gldh } A \geq r + 1 \geq n$ .

Furthermore  $\text{pd } A + \text{depth } A = \text{gldh } A \geq n$ . As  $\text{pd } A = 0$  it is  $\text{depth } A \geq n$  and by  $\text{depth } A \leq \dim A = n$  also  $\text{depth } A = \dim A = n$ . So we have  $\text{gldh } A = n = \dim A$  and  $A$  is a Cohen–Macaulay–ring.

Let  $\mathfrak{p}_i$  be the prime ideals of  $\text{Ass } A$ , all minimal. Then it holds  $\mathfrak{m} \not\subseteq \mathfrak{m}^2 \cup \bigcup_i \mathfrak{p}_i$ . So a  $g \in \mathfrak{m} - \mathfrak{m}^2$ , exists, which is not a zero-divisor in  $A$  and gives an exact sequence

$$0 \rightarrow A \xrightarrow{\cdot g} A \rightarrow A/gA \rightarrow 0$$

We call  $A' = A/gA$ .

Consider the change-of-ring spectral sequence (Eisenbud [1][Exercise A3.45 b, p. 682]:

$$E_{pq}^2 = \text{Tor}_q^{A'}(\text{Tor}_p^A(A', k), k) \Rightarrow \text{Tor}_{p+q}^A(k, k) \quad (4)$$

Because of  $0 \rightarrow A \xrightarrow{\cdot g} A \rightarrow A' \rightarrow 0$  it is

$$\text{Tor}_0^A(A', k) = k \quad (5)$$

$$\text{Tor}_1^A(A', k) = k \quad (6)$$

$$\text{Tor}_p^A(A', k) = 0 \text{ f\"ur } p \geq 2 \quad (7)$$

So we have  $E_{pq}^2 = 0$  for all  $p \neq 0, 1$ .

From  $\text{Tor}_\nu^A(k, k) = 0$  for  $\nu > n$  it follows that  $d_{0q}^2 : E_{0q}^2 \rightarrow E_{1, q-2}^2$  is an isomorphism for  $q \geq n + 2$ . Could we prove that  $d_{0q}^2 = 0$  for all  $q$ , we would have  $\text{Tor}_q^{A'}(k, k) = 0$  for  $q \geq n$ .

Now write  $\mathfrak{m}/\mathfrak{m}^2 = k(x_1 + \mathfrak{m}^2) + \dots + k(x_m + \mathfrak{m}^2)$  with  $x_i + \mathfrak{m}^2$  a  $k$ -base of  $\mathfrak{m}/\mathfrak{m}^2$ . Furthermore set  $x_1 = g$ .

Let  $P_\bullet \rightarrow k$  a free minimal  $A$ -resolution of  $k$  and  $Q_\bullet \rightarrow k$  another one of  $A'$ -modules. Without restriction of generality  $P_0 = A$  and  $P_1 = A^m$  where  $d_1 : P_1 \rightarrow P_0$  is given by  $(a_1, \dots, a_m) \mapsto \sum_{i=1}^m a_i x_i$ .

Analogously one assumes  $Q_0 = A'$  and  $Q_1 = A'^{m-1}$  with  $d_1 : Q_1 \rightarrow Q_0$  given by  $(a'_2, \dots, a'_m) \mapsto \sum_{j=2}^m a'_j x_j + \mathfrak{m}^2$ .

By a simple calculation one sees, that one can choose  $E_{pq}^0 = P_p \otimes_A Q_q$  for the  $E_{pq}^2$  from above, and that this  $E_{pq}^2$  is formed by a horizontal filtration of  $E_{pq}^0$ .

Because of Lemma 1.1 one can assume  $d_i(P_i) \subseteq \mathfrak{m}P_{i-1}$  and  $d_i(Q_i) \subseteq \bar{\mathfrak{m}}Q_{i-1}$ . This holds for  $d_2^P$  resp.  $d_2^Q$ , too, where it is not obvious by the definition. In case of  $P_\bullet$  we have as kernel of  $d_1$  the elements  $a_1, \dots, a_m$  with  $\sum_{j=1}^m a_j x_j = 0$ . If we had, say,  $a_1 \in A^*$ , then it would be  $x_1 = -\sum_{j=2}^m a_j a_1^{-1} x_j$  contradicting the assumption that  $(x_i + \mathfrak{m}^2)_i$  is a  $k$ -base of  $\mathfrak{m}/\mathfrak{m}^2$ . The case of  $d_2^Q$  can be concluded in the same way.

It follows from the horizontal filtration of the spectral sequence and definition 1.1, that an element  $\xi_{pq} \in E_{pq}^2$  can be represented by a pair  $x_{pq}, x_{p+1, q-1}$  for which

$$d_I(x_{pq}) = 0 \quad (8)$$

$$d_{II}(x_{pq}) + d_I(x_{p+1, q-1}) = 0 \quad (9)$$

holds. Here and in the following text  $x_{ij}$  (or  $y_{ij}$  and so on) stands for an element of  $E_{ij}^0$ . The derivations are  $d_I : E_{pq}^0 \rightarrow E_{p-1, q}^0$  and  $d_{II} : E_{pq}^0 \rightarrow E_{p, q-1}^0$ .

The pair  $x_{pq}, x_{p+1, q-1}$  represents zero exactly when there is a pair  $y_{p, q+1}, y_{p+1, q}$ , so that

$$0 = d_I(y_{p, q+1}) \quad (10)$$

$$x_{pq} = d_{II}(y_{p, q+1}) + d_I(y_{p+1, q}) \quad (11)$$

holds. (The pair  $y_{p, q+1}, y_{p+1, q}$  is from  $Z_1^{p-1}$  in the notation of equation (2) where the  $q+1$  resp.  $q$  resp.  $q-1$  from our  $E_{pq}^0$  stands for  $p-1$  resp.  $p$  resp.  $p+1$  in the cited equation. See also [2][p.87, eq. (1), p.88 eq. (2)] — one has to mirror the case described there on a line going diagonal from upper left to lower right).

We now show that for  $x_{10} \in P_1 \otimes_A Q_0 = Q_0^m$  with  $d_I(x_{10}) = 0$  and  $x_{10} \in \mathfrak{m}(P_1 \otimes_A Q_0)$  we always have  $x_{10} = d_I(y_{20})$  with a certain  $y_{20}$ .

First remember that  $k = \text{Tor}_1^A(k, A') = h_1(P_\bullet \otimes_A Q_0) = \ker(d_I)_{10} / \text{im}(d_I)_{20}$ .

Let now be

$$x_{10} \in Q_0 \otimes P_1 = Q_0^m = A'^m$$

given by

$$x_{10} = (\bar{v}^1, \dots, \bar{v}^m)$$

with  $\bar{v}_i \in Q_0 = A'$ , so that  $d_I(x_{10}) = 0$  is equivalent to  $\sum_{j=2}^m \bar{x}_j \bar{v}^j = 0$ . The component  $\bar{v}^1 \in A'$  can be chosen arbitrary.

So for a certain  $a_1 \in A$  we have

$$x_2 v^2 + \dots + x_m v^m + x_1 a^1 = 0$$

and, as  $P_\bullet$  is exact, also  $(a^1, v^2, \dots, v^m) \in d_I^P(P_2)$ . So we have

$$\ker(d_I)_{10} / \text{im}(d_I)_{20} = A' / \mathfrak{a}.$$

To see this in full clarity, one takes the map  $\phi : A' \rightarrow P_1 \otimes_A A' = P_1 \otimes_A Q_0$  with  $a' \mapsto (a', 0, \dots, 0)$ . By the preceding, we have  $\phi(A') \subseteq \ker(d_I)_{10}$  and even  $\pi \circ \phi : A' \rightarrow \ker(d_I)_{10} / \text{im}(d_I)_{20}$  surjective. So it is, as contended,  $A' / \mathfrak{a} = \ker(d_I)_{10} / \text{im}(d_I)_{20}$  with a suitable ideal  $\mathfrak{a}$ .

On the other hand we have

$$k = \text{Tor}_1^A(k, A') = \ker(d_I)_{10} / \text{im}(d_I)_{20} = A' / \mathfrak{a} = A' / \mathfrak{m}' = k.$$

If now  $x_{10} = (\bar{v}^1, \dots, \bar{v}^m)$  with  $d_I(x_{10}) = 0$  and we have  $v^1 + \mathfrak{m}Q_0 = 0$  by our second assumption, then  $x_{10} = 0$  as element of  $\text{Tor}_1^A(k, A')$ . So we can conclude that in these circumstances  $x_{10} = d_I(y_{20})$ .

Now consider  $d_{0q}^2 : E_{0q}^2 \rightarrow E_{1,q-2}^2$ . Here we have as representative of  $\xi_{0q}$  a pair  $x_{0q}, x_{1,q-1}$ . The representative of  $d_{0q}^2(\xi_{0q})$  is the pair  $d_{II}(x_{1,q-1}) = x_{1,q-2}, x_{2,q-3} = 0$ .

Now it is  $d_{II}(x_{1,q-1}) \in \mathfrak{m}(Q_{q-2} \otimes_A P_1)$ . Furthermore it is  $d_I(x_{1,q-2}) = 0$ . Now  $x_{1,q-2} \in Q_{q-2} \otimes_A P_1$  consists of as many components  $(x_{1,q-2})_\mu$  from  $Q_0 \otimes_A P_1 = A' \otimes_A P_1$  as given by the rank of  $Q_{q-2}$ . For any component the conditions assumed above for  $x_{10}$  are fulfilled. That is, if  $(x_{1,q-2})_\mu \in P_1 \otimes_A Q_0$  stands for such a component, then we have  $d_I((x_{1,q-2})_\mu) = 0$  and  $(x_{1,q-2})_\mu \in \mathfrak{m}(P_1 \otimes_A Q_0)$ . So every component is the image of a corresponding component from  $Q_{q-2} \otimes_A P_2$ .

Alltogether we have  $x_{1,q-2} = d_I(y_{2,q-2})$ . Now choosing  $y_{1,q-1} = 0$  it is  $d_{II}(y_{1,q-1}) + d_I(y_{2,q-2}) = x_{1,q-2}$  so that the pair  $(x_{1,q-2}, x_{2,q-3} = 0)$  represents the zero element of  $E_{1,q-2}^2$ . So it is finally  $d_{0q}^2 = 0$ .

Drawing all together, we have  $\text{gldh } A' < \infty$  and by induction, that  $A'$  is a geometrically regular local ring of dimension  $n - 1$ . So we have  $m = 1 + (n - 1) = n$  and

$$\mathfrak{m}/(\mathfrak{m}^2 + (g)) = (\bar{x}_2, \dots, \bar{x}_n) \quad (12)$$

So it is  $\mathfrak{m} = (g, x_2, \dots, x_n)$  and  $A$  is geometrically regular too.

## References

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